Design and analysis of a new singularity-free three-prismatic-revolute-cylindrical translational parallel manipulator

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Abstract: A new three-prismatic-revolute-cylindrical (three-PRC) translational parallel manipulator (TPM) with orthogonally arranged fixed actuators is proposed in this paper. The mobility of the manipulator is analysed via screw theory. The inverse kinematics, forward kinematics, and velocity analyses are performed and the singularities and isotropic configurations are identified afterwards. Moreover, the mechanism design rules producing a singularity-free manipulator have been generated. Under different cases of physical constraints subject to mechanical joints, the reachable workspace of the manipulator is geometrically determined and compared. In particular, it is illustrated that the manipulator has a regular shape workspace with a maximum cuboid defined as the usable workspace inscribed and one isotropic configuration involved. Furthermore, to obtain a large usable workspace, the architecture design of a three-PRC TPM is carried out and the singularity property within the usable workspace is verified. Simulation results show that there are no singular configuration within the workspace, which reveal the validity of design rules for a singularity-free three-PRC TPM.

Keywords: parallel manipulators, mechanism design, kinematics, workspace, singularity

1 INTRODUCTION

In the field of robotics, parallel manipulators have received extensive attentions over the past two decades. A parallel manipulator typically consists of a mobile platform that is connected to a fixed base by several limbs or legs in parallel as its name implies. The rigidity and payload carrying capacity of this kind of manipulator are better than those of a conventional serial manipulator; therefore it is more suitable for applications where high precision, stiffness, velocity, and heavy loads are required. Most six-degrees-of-freedom (six-DOF) parallel manipulators are based on the Gough–Stewart platform architecture because of the aforementioned advantages. However, six DOF is not always needed in many practical applications.

Therefore, in recent years, different parallel manipulators with fewer than six DOF called limited-DOF manipulators which maintain the inherent advantages of parallel mechanisms and possess several other advantages such as reduction in total cost of the device, are considerably attracting the attentions of researchers. Many spatial limited-DOF parallel manipulators were designed and manufactured for various applications in the past, such as the famous DELTA robot with three translational and an additional rotational DOF [1] whose concept was realized in different configurations [2, 3]; the Orthoglide parallel robots with pure translational DOF [4], spherical mechanisms with pure rotational motions [5, 6], three-PRS parallel manipulators [7, 8], and the HALF parallel robot [9] with mixed DOF etc.

Among these architectures, translational parallel manipulators (TPMs) have potential applications in cases demanding a pure translational motion such as a motion simulator, a positioning tool of an assembly
Several other types of architectures were proposed to achieve pure translational motions with different theoretical approaches, like the three-UPU, three-RUU, and three-PUU mechanisms [10], three-RC structure [11], three-RPC architecture [12], three-CRR manipulator [13, 14], etc. Here the notation of R, P, U, C, and S denotes the revolute joint, prismatic joint, universal joint, cylindrical joint, and spherical joint, respectively. In addition, the recent advances in the systematic type synthesis of TPMs could be found in literatures [15–19].

The three-prismatic-revolute-cylindrical (three-PRC) kinematical structure was first presented in reference [20] with the three P joint axes perpendicular to the fixed base, and a three-PRC TPM with adjustable layout angles of actuators was proposed by the authors [21]. In the current paper, a new three-PRC architecture TPM with orthogonally arranged fixed actuators is proposed and investigated to achieve three pure translational DOF. First of all, the fixed actuators make it possible that the moving components of the manipulator do not bear the load of actuators, which enables large powerful actuators to drive relatively small structures, and consequently facilitates the design of manipulators with faster, stiffer, and stronger characteristics. Moreover, the orthogonal arrangement of the three linear actuators results in a manipulator with a cubic shape of the workspace. Additionally, being an overconstrained mechanism, the proposed three-PRC TPM is constructed using fewer links and joints, and possesses a much simpler structure than most of the existing TPMs, which leads to an extensive reduction in cost and complexity of the device.

In addition, it is well known that the notable drawbacks of parallel manipulators involve limited workspace and complicated singularity problems [22–26]. For a parallel manipulator, the singular configuration results in a loss of the controllability and degradation of the natural stiffness of the manipulator. From this point of view, it is necessary to design a singularity-free parallel manipulator with a large workspace range so as to enhance its working capabilities.

A preliminary study of the proposed three-PRC TPM was presented by the authors in reference [27]. In this paper, the analysis has been revised in depth with the primary goal of achieving a singularity-free manipulator. And the remainder of the paper is organized in the following way. After an architectural description of the designed three-PRC TPM, the manipulator mobility is analysed in section 2. The closed-form solutions for both the inverse and forward kinematics problems are derived, and the velocity analysis is performed in section 3. Then in section 4, four types of singular configurations are investigated, where the mechanism design resulting in a singularity-free manipulator is presented. Additionally, in section 5, the isotropic configurations are determined analytically. Under various conditions of physical constraints, the reachable workspace is generated and compared in section 6. And as an example, the architecture design of a singularity-free three-PRC TPM is performed in section 7 in order to achieve a large usable workspace with several case studies provided, and the singularity verification is carried out via an adopted performance measure. Finally, some remarkable conclusions including considerations of a real machine construction are given in section 8.

2 DESCRIPTIONS AND MOBILITY ANALYSIS OF THE MANIPULATOR

2.1 Kinematical architecture description

A CAD model and a schematic diagram of the new three-PRC TPM are shown in Figs 1 and 2, respectively. It consists of a mobile platform, a fixed base, and three limbs with identical kinematic structure. Each limb connects the fixed base to the mobile platform by a P joint, a R joint, and a C joint in sequence, where the P joint is driven by a linear actuator assembled on the fixed base. Thus, the mobile platform is attached to the base by three identical PRC linkages.

Fig. 1 A three-PRC TPM
that limb. Moreover, in order to generate a cuboid like workspace of the manipulator as illustrated in the following discussions, the three P joints are arranged in an orthogonal manner.

2.2 Mobility analysis

The general Grübler–Kutzbach criterion is valuable for mobility analysis of many parallel manipulators, however, it is difficult to apply this criterion directly to the mobility analysis of overconstrained limited-DOF parallel manipulators. For example, the number of DOF of a three-PRC TPM given by the general Grübler–Kutzbach criterion is

$$F = \lambda(n - j - 1) + \sum_{j=1}^{j} f_i = 6 \times (8 - 9 - 1) + 12 = 0$$

where \(\lambda\) represents the dimension of the task space, \(n\) is the number of links, \(j\) is the number of joints, and \(f_i\) denotes the DOFs of joint \(i\). Another drawback of the general Grübler–Kutzbach criterion is that it can only derive the number of DOF of some mechanisms but cannot obtain the properties of the DOF, i.e. whether they are translational or rotational DOF.

On the contrary, the mobility of a three-PRC TPM can be effectively analysed by resorting to screw theory \([18, 28, 29]\), or the mathematical group theory \([15]\), etc. For a limited-DOF parallel manipulator, the motion of each limb that can be treated as a twist system is guaranteed under some exerted structural constraints, which is termed a wrench system. The wrench system is the reciprocal screw system of the twist system of the limb, and a wrench is said to be reciprocal to a twist if the wrench produces no work along the twist. The mobility of the manipulator is then determined by the combined effect of wrench systems of all the limbs.

Concerning a three-PRC TPM presented here, the twist system of each limb is a four-order screw system, and it is not difficult to derive the wrench system that is a reciprocal screw system of order 2, which exerts two constraint couples to the mobile platform with their axes perpendicular to the axis of the R joint. The wrench system of the mobile platform, that is the linear combination of wrench systems of all the three limbs, is a system of order 3 because the three wrench systems of order 2 consist of six couples which are linearly dependent and form a screw system of order 3. Since the direction of each R joint axis satisfies the conditions described earlier, i.e. it is invariable, the wrench systems restrict three rotations of the mobile platform with respect to the fixed base at any instant. Thus leading to a TPM.

It should be noted that the mobility of a three-PRC TPM can also be determined by adopting other methods, such as the new mobility criterion proposed in reference \([30]\).

3 KINEMATIC MODELLING

3.1 Inverse kinematic modelling

The purpose of the inverse kinematics problem is to solve the actuated variables from a given position of the mobile platform.

The geometry of one typical kinematic chain is depicted in Fig. 3. To facilitate the analysis, as shown in Figs 2 and 3, a fixed Cartesian reference frame \(O(x, y, z)\) is assigned at the initial position of point \(P\), which is the intersection of the three C joint axes, with the \(x\), \(y\), and \(z\) axes parallel to the axis of P joint \(A_1\), \(A_2\), and \(A_3\), respectively. In addition, the \(i\)th limb \(C_iB_i\) \((i = 1, 2, 3)\) of length \(a\) is connected to the mobile platform at point \(B_i\) which is a point on the axis of the \(i\)th C joint. \(B_0^i\) denotes a point on the mobile platform that is coincident with the initial position of \(B_i\), and the distance between points \(B_0^i\) and \(P\) is denoted by \(b\). Due to the connection with the slider of the \(i\)th P joint,
the R joint $C_i$ is restricted to move along the guide way of the P joint $A_i$. Point $C_i^0$ denotes the initial position of $C_i$ on the guide way.

Generally, the position and orientation of the mobile platform with respect to the reference frame can be described by a position vector $p = [x\ y\ z]^T = OP$ of the reference point $P$, and a $3 \times 3$ rotation matrix $R$. Since the mobile platform of a three-PRC TPM possesses only a translational motion, $R$ becomes an identity matrix.

Referring to Fig. 3, a vector-loop equation can be written for the $i$th ($i = 1, 2, 3$) limb as follows

$$a t_i = L_i - d_i u_i$$

(2)

with

$$L_i = p + b_i^0 + s_i k_i - c_i^0$$

(3)

where $t_i$ is the unit vector along $C_i B_i$, $d_i$ represents the linear displacement of the $i$th actuated joint with $u_i$ denoting the unit vector along the actuated direction, $s_i$ is the stroke of the $i$th C joint, $k_i$ denotes the unit vector parallel to the axes of the C and R joints of limb $i$, $b_i^0$, and $c_i^0$ represent the position vectors of points $B_i^0$ and $C_i^0$, respectively, at the initial position and the constant vectors can be expressed as follows

$$u_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(4a)

$$k_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad k_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad k_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

(4b)

$$b_i^0 = \begin{bmatrix} 0 \\ -b_i \\ 0 \end{bmatrix}, \quad b_2^0 = \begin{bmatrix} 0 \\ 0 \\ -b_2 \end{bmatrix}, \quad b_3^0 = \begin{bmatrix} -b \end{bmatrix}$$

(4c)

$$c_i^0 = \begin{bmatrix} a \\ b \end{bmatrix}, \quad c_2^0 = \begin{bmatrix} -a \\ b \end{bmatrix}, \quad c_3^0 = \begin{bmatrix} 0 \\ -a \end{bmatrix}$$

(4d)

Substituting equation (3) into equation (2) and dot-multiplying both sides of the expression by $k_i$, the derivation of $s_i$ is obtained, that is

$$s_i = k_i^T (e_i^0 - b_i^0 - p)$$

(5)

In addition, substituting equations (4b) to (d) into equation (5), yields

$$s_1 = y, \quad s_2 = z, \quad s_3 = x$$

(6)

Dot-multiplying equation (2) with itself and rearranging the items, one has

$$d_i^2 = -2d_i^2 u_i^T L_i + L_i^T L_i - a^2 = 0$$

(7)

Then, solving equation (7) leads to solutions for the inverse kinematics problem

$$d_1 = a + x \pm \sqrt{a^2 - x^2}$$

(8a)

$$d_2 = a + y \pm \sqrt{a^2 - y^2}$$

(8b)

$$d_3 = a + z \pm \sqrt{a^2 - z^2}$$

(8c)

It can be observed that there exist two solutions for each actuated variable, hence there are totally eight possible solutions for a given mobile platform position. For instance, given a position $p = [0\ 0\ 0]^T$ of the mobile platform for a three-PRC TPM with architecture parameters of $a = 100$ cm and $b = 50$ cm, the actuated variables calculated through the inverse kinematics solutions with pure negative and pure positive square roots in equation (8) are illustrated in Figs 4(a) and (b), respectively. To avoid the case as described in Fig. 4(b), in which the links are overlapped with the C joints that may probably not occur in practice due to the rotary limits of R and C joints, only the negative square roots in equation (8) are selected to yield a unique solution for practical applications.

### 3.2 Forward kinematic modelling

Given a set of actuated inputs, the position of the mobile platform is solved by the forward kinematics.

Rearranging equation (8) and taking the square of them, allows one to generate

$$(x + a - d_1)^2 + z^2 = a^2$$

(9a)

$$x^2 + (y + a - d_2)^2 = a^2$$

(9b)

$$y^2 + (z + a - d_3)^2 = a^2$$

(9c)

Equation (9) represents three cylindrical surfaces, and the intersection of them forms the solutions to the forward kinematics problem, which can be derived via the Sylvester dialytic elimination method as follows.

First, writing equations (9a) and (9c) into second-degree polynomials in $z$, yields

$$z^2 + m_1 z + m_3 = 0$$

(10)

$$z^2 + m_2 z + m_3 = 0$$

(11)

where $m_1 = (x + a - d_1)^2 - a^2$, $m_2 = 2(a - d_3)$, and $m_3 = y^2 + (a - d_3)^2 - a^2$.

Taking equation (11)−(10) and equation (11) $\times m_1 - (10) \times m_3$, respectively, and rewriting the two equations into the following matrix form

$$\begin{bmatrix} m_2 & m_3 - m_1 \\ m_1 - m_3 \\ m_1 m_2 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(12)

which represents a system of two linear equations in $z$ and 1. Then, the following equation can be obtained
by equating the determinant of the coefficient matrix to zero

\[ m_1 m_2^2 + (m_1 - m_3)^2 = 0 \] (13)

that can be expanded into

\[ x^4 + n_1 x^3 + n_2 x^2 + n_3 x + n_4 = 0 \] (14)

where \( n_1, n_2, n_3, \) and \( n_4 \) are all polynomials only in \( y \).

In addition, writing equation (15) into the form

\[ x^2 + m_4 = 0 \] (15)

where \( m_4 = (y + a - d_4)^2 - a^2 \).

Taking equation (14) \(-\times\ x^2\), one can obtain that

\[ n_1 x^3 + (n_2 - m_4) x^2 + n_3 x + n_4 = 0 \] (16)

Similarly, taking equation \((\times\ x - (15)\times (x^3 + n_1 x^2))\), yields

\[ (n_2 - m_4) x^3 + (n_3 - m_4 n_1) x^2 + n_4 x = 0 \] (17)

Additionally, multiplying both sides of equation (15) by \( x \), leads to

\[ x^3 + m_4 x = 0 \] (18)

Thus, equations (15) to (18) can be considered as four linear homogeneous equations in the four variables of \( x^3, x^2, x \), and \( 1 \). The characteristic determinant is

\[
\begin{vmatrix}
  n_1 & n_2 - m_4 & n_3 & n_4 \\
  n_2 - m_4 & n_3 - m_4 n_1 & n_4 & 0 \\
  1 & 0 & m_4 & 0 \\
  0 & 1 & 0 & m_4 \\
\end{vmatrix} = 0
\] (19)

Expanding equation (19) results in an eighth-degree polynomial in one variable of \( y \). Once \( y \) is found, a unique value of \( x \) can be calculated by setting the first-degree greatest common divisor between equations (14) and (15) to be zero, and a unique value of \( z \) can then be computed by solving equation (12). There are totally eight possible solutions for the forward kinematics with at most eight positions of the mobile platform.

However, as illustrated in the following discussions, the manipulator workspace is restricted partly by the physical constraints imposed by the stroke limits of \( \text{C} \) joints and motional range limits of linear actuators, that is

\[ s_{\text{min}} \leq s_i \leq s_{\text{max}} \] (20a)
\[ d_{\text{min}} \leq d_i \leq d_{\text{max}} \] (20b)

for \( i = 1, 2, 3 \). Combining equation (6) with equation (20a), one can derive that

\[ s_{\text{min}} \leq x, y, z \leq s_{\text{max}} \] (21)

Therefore, although there are multiple solutions for the forward kinematics problem, it can be revealed that only one real solution lies within the workspace range, which is taken into consideration for practical applications.

### 3.3 Velocity analysis

Substituting equation (3) into equation (2) and differentiating the result with respect to time, leads to

\[
\dot{d}_i u_i + a \omega_i \times \mathbf{t}_i = \dot{\mathbf{p}} + \dot{\mathbf{v}}_i \mathbf{k}_i
\] (22)

where \( \omega_i \) is the angular velocity vector of the \( i \)th limb \( \text{C}_i \text{B}_i \) with respect to the reference frame, and \( \dot{\mathbf{p}} = [\dot{x} \ \dot{y} \ \dot{z}]^T \) denotes the vector of linear velocities for the mobile platform.
To eliminate the passive variable $\omega_i$, both sides of equation (22) are multiplied by $t_i$, this gives

$$t_i^T u_i d_i = t_i^T p$$

(23)

Writing equation (23) for $i = 1, 2,$ and $3$, respectively, yields three scalar equations, which can be rewritten into the matrix form

$$Bq = A\dot{p}$$

(24)

where

$$B = \begin{bmatrix} t_i^T u_i & 0 & 0 \\ 0 & t_i^T u_2 & 0 \\ 0 & 0 & t_i^T u_3 \end{bmatrix}$$

(25a)

$$A = \begin{bmatrix} t_i^T \\ t_i^T \\ t_i^T \end{bmatrix}$$

(25b)

are called the inverse and forward kinematics Jacobian matrices, respectively, and $\dot{q} = [d_1, d_2, d_3]^T$ represents the vector of actuated joint rates.

Solving $t_i$ from equation (2), then substituting $t_i$ and the constant vectors expressed by equation (4) into equation (25), allows the generation of

$$B = \frac{1}{a} \begin{bmatrix} x + a - d_1 & 0 & 0 \\ 0 & y + a - d_2 & 0 \\ 0 & 0 & z + a - d_3 \end{bmatrix}$$

(26a)

$$A = \frac{1}{a} \begin{bmatrix} x + a - d_1 & 0 & z \\ x & y + a - d_2 & 0 \\ 0 & y & z + a - d_3 \end{bmatrix}$$

(26b)

When the manipulator is away from singularities, the following velocity equation can be derived from equation (24).

$$\dot{q} = J\dot{p}$$

(27)

where

$$J = B^{-1} A$$

(28)

$$= \begin{bmatrix} 1 & 0 & \frac{z}{x + a - d_1} \\ \frac{y + a - d_2}{x} & 1 & 0 \\ 0 & \frac{y}{z + a - d_3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e_i^T \\ e_j^T \\ e_k^T \end{bmatrix}$$

is defined as the Jacobian matrix of a three-PRC TPM, which relates output velocities to the actuated joint rates.

4 SINGULARITY-FREE MANIPULATOR DESIGN

In what follows, all kinds of singularities for the three-PRC TPM are identified and eliminated via the approach of a proper mechanism design so as to generate a singularity-free manipulator.

1. The first kind of singularity, which is also called the inverse kinematics singularity, occurs when $B$ is not of full rank and $A$ is invertible, i.e. $\det(B) = 0$ and $\det(A) \neq 0$.

From equation (26a), it is the case when one or more of the following equations hold

$$x + a - d_1 = 0$$

(29a)

$$y + a - d_2 = 0$$

(29b)

$$z + a - d_3 = 0$$

(29c)

In addition, from equation (25a), it is noticed that this is the situation when $t_i \perp u_i$, for $i = 1, 2,$ or $3$, i.e. the directions of one or more of the legs are perpendicular to the axial directions of the corresponding actuated joints. Under this case, the mobile platform loses one or more DOF, which always occurs on the boundary of the workspace.

Figure 5(a) illustrates an example of the inverse kinematics singularity in case of $C, B_i \perp A_i C_i$. In view of equation (6), $s_2 = z = a$ can be derived. Therefore, in order to eliminate this kind of singularity, the stroke limits of C joints should satisfy the following conditions

$$s_{\text{min}} > -a \quad \text{and} \quad s_{\text{max}} < a$$

(30)

2. The second kind of singularity, which is also termed as the direct kinematics singularity, occurs when $A$ is not of full rank while $B$ is invertible, i.e. $\det(B) \neq 0$ and $\det(A) = 0$.

From equation (26b), it is observed that this is the case when the following equation holds

$$(x + a - d_1)(y + a - d_2)(z + a - d_3) + xyz = 0$$

(31)

In view of equation (25b), one can deduce that the vectors $t_i$ ($i = 1, 2, 3$) become linearly dependent. Under such case, the manipulator gains one or more DOF even when all actuators are locked. Geometrically, there are three cases for this kind of singularity.

Case 1. Two legs are parallel to each other. Without loss of generality, assume that $t_1 \parallel t_2$. Taking into account the expression of $t_i$ in equation (25b) and the conditions for two spatial vectors parallel to each other from the geometry knowledge along with the consideration of equation (31), allows
the generation of the manipulator configuration as shown in Fig. 5(c), where \( C_iB_i \parallel C_2B_2 \). It is obvious that the condition in equation (29b) is also satisfied since \( C_2B_2 \perp A_2C_2 \), thereby this is not a direct kinematics singularity but a combined singularity discussed later.

**Case 2.** Three legs are parallel to each other. It can be deduced from equation (25b) that this will never happen since when any two legs are parallel to each other, the third leg is always perpendicular to them.

**Case 3.** Three legs are coplanar. Again, via the geometry knowledge also considering equation (31), it can be derived that

\[
d_1 = d_2 = d_3 \quad \text{and} \quad x = y = z = \frac{d_i - a}{2} \quad (32)
\]

and such a singular configuration is shown in Fig. 5(b), where the six points of \( B_i \) and \( C_i \) (\( i = 1, 2, 3 \)) lie on a common plane. In order to eliminate this type of singularity from the manipulator workspace, it can be set as \( d_i = d_{\text{min}} \), for \( i = 1, 2, \) and 3. Then, the minimum value of the C joints stroke should be designed as follows by taking into account equation (20)

\[
s_{\text{min}} > \frac{d_{\text{min}} - a}{2} \quad (33)
\]

3. The third kind of singularity, which is also called the combined singularity, occurs when \( B \) and \( A \) become simultaneously not invertible, i.e. \( \det(B) = 0 \) and \( \det(A) = 0 \).

Under this type of singularity, the mobile platform can undergo finite motions even when the actuators are locked, or equivalently, it cannot resist to forces or movements in one or more directions even if all actuators are locked. And some finite motions of the actuators give no motions of the mobile platform.

It can be deduced that the combined singularity occurs if and only if one or more conditions in equation (29) and the condition expressed by equation (31) are met at the same time. Depending on the number of satisfied conditions expressed by equation (29), there are also three cases:

**Case 1.** Only one condition in equation (29) is satisfied, i.e. only one leg is perpendicular to
its corresponding actuated joint axis. For example, when \( C_i B_2 \perp A_i C_i \), the singular configuration is illustrated in Fig. 5(c) with \( C_i B_2 \parallel C_i B_1 \), which meets the conditions in equations (29b) and (31).

**Case 2.** Only two conditions in equation (29) are satisfied. For example, the singular configuration with geometry relationships of \( C_i B_1 \perp A_i C_i \), \( C_i B_2 \perp A_i C_i \), and \( C_i B_1 \parallel C_i B_2 \) is depicted in Fig. 5(d), which meets the conditions expressed by equations (29a), (29b), and (31).

**Case 3.** Three conditions in equation (29) are satisfied. Combining equation (29) and equation (31), the derivations obtained are

\[
d_1 = d_2 = d_3 = a \quad \text{and} \quad x = y = z = 0 \quad (34)
\]

Under such case, in view of equation (2), it can be deduced that equation (34) implies \( a \mathbf{e} = 0 \), which is never true for the current three-PRC TPM. Thus, this case will never occur.

From the above discussions, it is observed that the conditions expressed by equation (30) should be satisfied as well so as to avoid combined singularities.

4. Besides the three types of singularities discussed above, the rotational singularity for a TPM may occur when the mobile platform of a TPM can rotate instantaneously. This concept has been extended to the constraint singularity of limited-DOF parallel manipulators, which arises when the kinematic chains of a limited-DOF parallel manipulator cannot constrain the mobile platform to the planned motion any more. As far as a three-PRC TPM is concerned, it has been shown based on the screw theory and analytical approach [29] that the mobile platform cannot rotate at any instant, thus there is no rotational singularity for the three-PRC TPM.

As a consequence, for the sake of achieving a singularity-free three-PRC TPM, the conditions must hold

\[
\sigma = \mathbf{e}_i^T \mathbf{e}_i = 1 \quad (36a)
\]

\[
\mathbf{e}_i^T \mathbf{e}_j = 0 \quad \text{for} \ i \neq j \quad (36b)
\]

for \( i, j = 1, 2, 3 \).

In view of equations (28) and (36), it can be deduced that \( x = y = z = 0 \), i.e. \( \mathbf{p} = [0 \ 0 \ 0]^T \), which is the configuration in case of the mobile platform at its home position.

## 6 WORKSPACE DETERMINATION

It is well known that compared with their serial counterparts, parallel manipulators have relatively limited workspace. Thus the workspace of a parallel manipulator is one of the most important aspects to reflect its working capacity, and it is necessary to analyse the shape and volume of the workspace for enhancing applications of parallel manipulators. Furthermore, for a given set of dimensions of the manipulator, it is of particular interest to determine how the physical constraints subject to stroke limits of C joints and motion limits of linear actuators affect the manipulator workspace. The reachable workspace of a three-PRC TPM is defined as the space that can be reached by the reference point \( P \).

Actually, in many industrial applications, the workspace of a manipulator is usually in a cuboid or a cylinder shape. For a three-PRC TPM, the desired workspace is described as a maximum cuboid inscribed within the reachable workspace that is defined as the usable workspace, in which most of the practical applications will be performed.

Assume that the motion of linear actuators is constrained within the range of \([-D/2, D/2]\), and the stroke range of each C joint is restricted within \([-S/2, S/2]\). In view of equation (21), one has

\[
-\frac{S}{2} \leq x, y, z \leq \frac{S}{2} \quad (37)
\]

Considering equation (9) along with equation (37), one can deduce that the workspace of each limb is a set of cylinder segments with radius \( a \) and height \( S \) once the range of \( d_i \) is assigned. The intersection of the three limbs’ workspace forms the reachable workspace of the manipulator as illustrated in Fig. 6, and the workspace volume can also be calculated at the same time. It should be noticed that the workspace is irrespective of the parameter \( b \).

Assume that the stroke of each linear actuator be \( D = 40 \text{ cm} \). In view of equations (30) and (33), the architectural parameters of a three-PRC TPM are designed to be \( a = 100 \text{ cm} \) and \( b = 50 \text{ cm} \). The reachable workspace is partially constrained by the motion...
limit $D$ of linear actuators, and in part restricted by the stroke limit $S$ of C joints. The TPM workspace is generated and shown in Fig. 7 in three cases of $S = 50$ cm, $S = 40$ cm, and $S = 30$ cm, respectively.

From Fig. 7(a), it is observed that when $S > D$, the workspace is solely determined by the motion range of linear actuators. When $S = D$, it is shown in Fig. 7(b) that the workspace is determined by the combined effects of the stroke limits of C joints and the motion limits of linear actuators. And in the case of $S < D$, the workspace is determined only by the stroke limits of C joints as illustrated in Fig. 7(c), which is a cuboid with the edge length $S$. Moreover, it can be seen that the manipulator has the largest reachable workspace in the case of $S > D$ as shown in Fig. 7(a).

7 CASE STUDIES

In this section, the architecture design of a singularity-free three-PRC TPM is performed in order to achieve a maximum inscribed usable workspace, and the singularities of the manipulator within the workspace are verified with several case studies illustrated.

7.1 Architectural parameters selection

The cubic shape usable workspace of the three-PRC TPM is also shown in Fig. 7. It is observed that the size of the usable workspace in Fig. 7(a) is equal to that in Fig. 7(b), and both of them are larger than that in Fig. 7(c). Generally, the manipulator will perform tasks within the usable workspace, while the remainder of the reachable workspace can be used for additional work such as the change of the end-effector, etc. Thus, in order to generate a larger workspace, the stroke limit of C joints is to be chosen as $S > D$. Under this situation, the determination procedure of a usable workspace with the edge length of $W$ is described in Fig. 8, from which one can derive that

\[
D = W + 2c 
\]  
(38a)

\[
c = a - \sqrt{a^2 - \left(\frac{W}{2}\right)^2} 
\]  
(38b)

Combining equation (38a) with equation (38b), the result obtained is

\[
a = \sqrt{a^2 - \left(\frac{W}{2}\right)^2} = \frac{D - W}{2} 
\]  
(39)

Then, solving equation (39), allows the generation of

\[
W = \frac{D}{2} - a \pm \sqrt{4a^2 + 4Da - D^2} 
\]  
(40)

Only the positive square root in equation (40) is considered since the negative one has no meaning in practice. The motion range of linear actuators is assigned to be $D = 40$ cm. Then, the relationship between $a$ and $W$ is illustrated in Fig. 9. It is observed that the size of usable workspace increases as $a$ increases. However, the increment is not very much after $a > 80$ cm. For the purpose of comparison, three cases of $a = 60$ cm, $a = 80$ cm, and $a = 100$ cm are taken into account. In view of the design rules in equations (30) and (33) and considering the relationship of $S > D$, the stroke limit of C joints is designed to be $S = 50$ cm. Another parameter is designed as $b = 50$ cm.
Substituting the three sets of parameters of \( a \) (\( a = 60, 80, \) and 100 cm) and \( D = 40 \) cm into equation (40), we can obtain \( W = 34.83, 35.92, \) and 36.62 cm, respectively as shown in Table 1. It is observed that when the value of \( a \) increases from 60 to 80 cm, the size of the usable workspace is enlarged by 1.09 cm. Whereas when \( a = 80 \) cm is increased by 20 cm again, the increment in the usable workspace edge length is only 0.70 cm. Hence, in order to make a compromise between the size of usable workspace and the compactness of the manipulator, \( a = 80 \) cm is selected.

The ratio of the usable workspace size to the motion range of actuated joints is \( \eta = 35.92/40 = 0.898 \), that is a relatively large value compared to other parallel manipulators.

### 7.2 Singularity verification

The conditioning index (CI, \( \mu \)), which is defined as the reciprocal of the condition number of Jacobian matrix (\( \kappa \)) [31], can be utilized to evaluate the distance to the

![Fig. 10](image)

The distributions of conditioning index in three typical planes of \( z = -W/2 \), \( z = 0 \), and \( z = W/2 \), for a three-PRC TPM with three sets of parameters.
singularity. That is

$$\mu = \frac{1}{\kappa}$$

(41)

where $\kappa = \|J\| \|J^{-1}\|$, with $\| \cdot \|$ denoting the 2-norm of the matrix. The condition index ranges in value from zero to one and measures the degree of ill-conditioning of the Jacobian matrix, i.e. nearness of the singularity. In case of $\mu = 0$, the manipulator is in a singular configuration, therefore, the larger the conditioning index, the farther the distance to the singularity.

For a three-PRC TPM with three sets of parameters of $a$ ($a = 60, 80, \text{and} 100 \text{ cm}$) and other aforementioned parameters, the conditioning index ($\mu$) in a plane at every height within the workspace can be calculated. For each case, the distributions of $\mu$ in three typical planes of $z = 0$ and $z = \pm W/2$ are shown in Fig. 10. It is observed that, for each set of parameters of the three-PRC TPM, $\mu$ has a maximum value when the mobile platform lies in the $z$-axis. In particular, $\mu$ arrives at the maximal value 1 in case of $x = y = z = 0$ as shown in Figs 10(b), (e), and (h), which represents an isotropic configuration, and $\mu$ decreases as the mobile platform approaches to the workspace boundary. The minimum value of $\mu$ or the three sets of parameters are shown in Table 1. It is seen that the minimum $\mu$ increases as $a$ increases. However, the increment of minimum $\mu$ decreases from 0.09 to 0.06 as the increasing of parameter $a$ with the same increment of 20 cm.

Moreover, in all of the three cases, every value of $\mu$ in any planes is far larger than 0, which indicates that there exists no singular configuration within the usable workspace of the manipulator. The simulation results validate in part the validity of singularity-free design rules for a three-PRC TPM as expressed by equations (30) and (33).

8 CONCLUSIONS

In the present paper, a new three-PRC TPM with actuators mounted in orthogonal directions has been proposed. One obvious advantage of the new TPM is that it possesses a much simpler structure than most of the existing TPMs, which means an extensive reduction in cost in building and controlling such a manipulator. Both the inverse and forward kinematics solutions are given and the velocity equation is presented. All kinds of singular and isotropic configurations are identified, and the mechanism design to eliminate any singularities is described. The architecture design of a singularity-free three-PRC TPM has been carried out to achieve a large usable workspace, i.e. a maximum inscribed cuboid inside the reachable workspace. Moreover, the singularity verification within the workspace is performed, and the simulation results reveal that there exists no singularity within the usable workspace where most practical applications will be performed, which also validate the derived singularity-free design rules.

Additionally, as an overconstrained mechanism, the problems of variable friction in passive joints and large reaction movement have to be considered to assure the mobility of the mobile platform for a three-PRC TPM. Otherwise, the mobile platform may not move or the manipulator cannot work if there are some kinematic errors. These issues can be solved by adding a revolute joint with its axis along the axial direction of the actuated prismatic joint, thus the actual structure of the limbs becomes a prismatic-revolute–revolute-cylindrical (PRRC) mechanism. This design choice results in a non-overconstrained manipulator and causes no impact on the mobility and kinematics of the original manipulator. Since the proposed TPM is an isotropic manipulator owning a cubic like singularity-free workspace involving one isotropic configuration, the manipulator has potential applications in quite a lot of fields requiring high accuracy, such as assembly, machining, and especially, in microscale manipulation. The results presented in this paper will be valuable for the development of a new TPM prototype.

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REFERENCES

1 Clavel, R. DELTA, a fast robot with parallel geometry. In Proceedings of 18th International Symposium on Industrial robots, Lausanne, Switzerland. 1988, pp. 91–100.


**APPENDIX**

**Notation**

- $a$: length of each limb
- $A$: forward kinematics Jacobian matrix
- $b$: size of the mobile platform
- $b_i^j, c_i^0$: constant vectors
- $B$: inverse kinematics Jacobian matrix
- $C$: cylindrical joint
- $d_i$: linear displacement of the $i$th actuator
- $d_{\text{min}}, d_{\text{max}}$: stroke limits of the linear actuator
- $d_i^j$: the $i$th actuated joint rate
- $D$: stroke size of the linear actuator
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J )</td>
<td>Jacobian matrix</td>
</tr>
<tr>
<td>( k_i )</td>
<td>unit vector parallel to the C joint axis of the ( i )th limb</td>
</tr>
<tr>
<td>( p )</td>
<td>position vector of the mobile platform</td>
</tr>
<tr>
<td>( p )</td>
<td>vector of linear velocities for the mobile platform</td>
</tr>
<tr>
<td>( P )</td>
<td>prismatic joint</td>
</tr>
<tr>
<td>( q )</td>
<td>vector of the actuated joint variables</td>
</tr>
<tr>
<td>( \dot{q} )</td>
<td>vector of the actuated joint rates</td>
</tr>
<tr>
<td>( R )</td>
<td>revolute joint</td>
</tr>
<tr>
<td>( R )</td>
<td>rotation matrix</td>
</tr>
<tr>
<td>( s_i )</td>
<td>stroke of the ( i )th C joint</td>
</tr>
<tr>
<td>( s_{\text{min}}, s_{\text{max}} )</td>
<td>stroke limits of the C joint</td>
</tr>
<tr>
<td>( S )</td>
<td>spherical joint</td>
</tr>
<tr>
<td>( S )</td>
<td>stroke size of the C joint</td>
</tr>
<tr>
<td>( t_i )</td>
<td>unit vector along the ( i )th limb</td>
</tr>
<tr>
<td>( u_i )</td>
<td>unit vector along the ( i )th actuating direction</td>
</tr>
<tr>
<td>( U )</td>
<td>universal joint</td>
</tr>
<tr>
<td>( W )</td>
<td>edge length of the usable workspace</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>condition number of the Jacobian matrix</td>
</tr>
<tr>
<td>( \mu )</td>
<td>conditioning index</td>
</tr>
<tr>
<td>( \omega_i )</td>
<td>vector of angular velocities for the ( i )th limb</td>
</tr>
</tbody>
</table>